The Alberta High School Mathematics Competition
PART I
November 20, 2018

1. The average of 1, a and b is equal to $2c$. The average of 1, a and c is equal to $2b$. The average of 1, b and c is equal to $2a$. The average of a, b and c is equal to

(A) 0   (B) $\dfrac{1}{4}$ (C) $\dfrac{1}{3}$ (D) $\dfrac{1}{2}$ (E) $\dfrac{3}{4}$

Solution:
We have $1 + a + b = 6c$, $1 + a + c = 6b$ and $1 + b + c = 6a$. Adding gives $3 + 2a + 2b + 2c = 6a + 6b + 6c$ implying $a + b + c = \frac{3}{4}$, hence, $\frac{a+b+c}{3} = \frac{1}{4}$. The answer is (B).

Alternative solution: Using the conditions in the question, $1 + a + b + c = 7a = 7b = 7c$, hence $a = b = c = \frac{1}{4}$ and therefore $\frac{a+b+c}{3} = \frac{1}{4}$.

2. Let $a, b > 0$ and suppose the line $y = b - ax$ has an x-intercept of 2. If the triangle enclosed by the lines $x = 0$, $y = 0$ and $y = b - ax$ has area equal to 4, the value of a must be

(A) $\frac{1}{2}$ (B) 1   (C) 2   (D) 4   (E) none of these

Solution:
Since the triangle with vertices $(0, 0)$, $(2, 0)$ and $(0, b)$ has area equal to 4 we must have $\frac{1}{2}(2)(b) = 4$ implying $b = 4$. An x-intercept of 2 implies $0 = 4 - a(2)$, thus $a = 2$. The answer is (C).

3. Let f be a function such that $f\left(\frac{x+1}{3}\right) = x^2 + x + 3$ for any real number x. The sum of all real numbers a such that $f\left(\frac{a}{9}\right) = 10$ is

(A) -63   (B) -3   (C) 3   (D) 6   (E) 63

Solution:
If you set $y = \frac{x+1}{3}$ then $x = 3y - 1$, hence $f(y) = (3y - 1)^2 + (3y - 1) + 3 = 9y^2 - 3y + 3$. Now, $f\left(\frac{a}{9}\right) = 10$ is equivalent to $a^2 - 3a - 63 = 0$. This equation has real roots having the sum 3. The answer is (C).

4. A sequence is defined by $a_0 = 0$ and $a_{n+1} = \dfrac{a_n - 3}{a_n - 2}$ for all integers $n \geq 0$. Then a_{2018} is

(A) 0   (B) 1   (C) $\frac{3}{2}$ (D) 3   (E) none of these

Solution:
We have $a_0 = 0$, $a_1 = \frac{3}{2}$, $a_2 = \dfrac{\frac{3}{2} - 3}{\frac{3}{2} - 2} = 3$, $a_3 = 0, \cdots$ Hence the sequence repeats after 3 steps and $a_{3k} = 0$ for all positive integers k. Thus $a_{2016} = 0$ and hence $a_{2018} = 3$. The answer is (D).

5. Pete has \$$x$ in his pocket. If he buys two apples using his \$$x$ he will have \$3 in change remaining. If he buys three bananas using his \$$x$ he will have \$2 in change remaining. Similarly, if he buys four apples and one banana using his \$$x$ he will also have \$2 in change remaining. What is the maximum number of bananas Pete can afford to buy?

(A) 7   (B) 10   (C) 14   (D) 18   (E) 21

Solution:

Let A be the price of one apple and B the price of one banana. We are given $x = 2A + 3$, $x = 3B + 2$, $x = 4A + B + 2$. Thus, $2A = 3B - 1$ and $4A = 2B$ implying $6B - 2 = 2B$ or $B = 0.5$. This gives $A = 0.25$ and $x = 3.5$. Thus, Pete can buy a maximum of 7 bananas. The answer is (A).

6. A student writes a multiple choice exam. The exam contains 16 questions and the score is 5 points for a correct answer, 2 points for no answer and 0 points for an incorrect answer. Each student also receives 20 bonus points. If a student scores 78 points, how many questions were answered correctly?

(A) 8   (B) 9   (C) 10   (D) 11   (E) not uniquely determined

Solution:

Let x be the number of correct answers and y the number of non-answered questions. Thus, $0 \le x + y \le 16$ and

$$20 + 5x + 2y = 78 \iff 5x + 2y = 58.$$

Since $5x = 2(29 - y)$, 5 should be a divisor of $29 - y$, hence $y \in \{4, 9, 14\}$. The only convenient value for y is 4 and hence the solution is $x = 10$, $y = 4$. This gives 10 correct answers, 2 wrong answers and 4 no answers. The answer is (C).

Alternative solution: The total for a perfect paper is 100. If the student didn't answer a questions and the answers for b questions were not correct, then
$$3a + 5b = 22$$
where $0 \le a + b \le 16$. It is clear that $b \le 4$ and since $5b \equiv 1 \pmod 3$ one obtains $b = 2$ and then $a = 4$. Therefore, the remaining 10 questions are correct.

7. How many triplets (m, n, k), of integers greater than one, exist such that $(m!) \cdot (n!) = (k!)$?
(Notice that $n! = 1 \times 2 \times 3 \times \cdots \times n$.)

(A) 0   (B) 1   (C) 2   (D)3   (E) more than 3

Solution:

Let n be any positive integer greater than 2. Setting $m = n! - 1$ and $k = n!$ one obtains $(n! - 1)! \cdot (n!) = (n!)!$ and hence the equation has infinitely many solutions. The answer is (E).

8. The sum of all positive integers smaller than 126 which are not divisible by 2 and not divisible by 3 is

(A) 2000   (B) 2394   (C) 2400   (D) 2646   (E) none of these

Solution:

Any number that is not divisible by 2 and not divisible by 3 has the form $6k + 1$ or $6k + 5$, where $0 \le k \le 20$.

Therefore the sum is

$$S = \sum_{k=0}^{20} (6k + 1) + \sum_{k=0}^{20} (6k + 5) = \sum_{k=0}^{k=20} (12k + 6) = 12(1 + 2 + 3 + \cdots + 20) + 6 \cdot 21 = 12 \cdot 21 \cdot 10 + 6 \cdot 21 = 2646$$

Alternative Solution:

The sum of all numbers smaller than 126 is

$$1 + 2 + 3 + \ldots + 125 = 125 \cdot 63$$

The sum of all even numbers smaller than 126 is

$$2 + 4 + 6 + \ldots + 124 = 2(1 + 2 + 3 + \cdots + 62) = 62 \cdot 63$$

The sum of all multiples of three smaller than 126 is

$$3 + 6 + \ldots + 123 = 3(1 + 2 + \cdots + 41) = 3 \cdot 41 \cdot 21$$

The sum of all multiples of six smaller than 126 is

$$6 + 12 + \ldots + 120 = 6(1 + 2 + \cdot + 20) = 6 \cdot 21 \cdot 10$$

Therefore, the sum is

$$125 \cdot 63 - 62 \cdot 63 - 3 \cdot 41 \cdot 21 + 6 \cdot 21 \cdot 10 = 2646$$

9. Let k be a fixed positive real number. Suppose $kx^2 \le 10$ has exactly five positive integer solutions. The number of positive integer solutions to $kx \le 1$ must be

   (A) one or two   (B) two or three   (C) three or four   (D) four or five   (E) none of these

   Solution:
   The given inequality may be written as follows:

   $$kx^2 \le 10 \iff x^2 \le \frac{10}{k} \iff x \in \left[-\sqrt{\frac{10}{k}}, \sqrt{\frac{10}{k}} \right]$$

   The inequality $kx^2 \le 10$ has exactly five positive integer solutions if and only if

   $$\sqrt{\frac{10}{k}} \in [5, 6) \iff \frac{10}{k} \in [25, 36) \iff \frac{1}{k} \in [2.5, 3.6).$$

   Sine the inequality $kx \le 1$ is equivalent to $x \le \frac{1}{k}$, the number of positive integer solutions is two if $\frac{1}{k} \in [2.5, 3)$ and three if $\frac{1}{k} \in [3, 3.6)$. The answer is (B).

10. Consider polynomials $p(x)$ of degree 5 with leading coefficient 1 such that $p(-x) = -p(x)$ for all real x. If the roots of $p(x)$ include -1 and $\sqrt{2}$, the value of $p(2)$ is

    (A) 0   (B) 6   (C) 12   (D) 30   (E) not uniquely determined

    Solution:
    From $p(-x) = -p(x)$ we have $p(0) = -p(0)$ implying $p(0) = 0$, thus $x = 0$ is a root. Further, $p(1) = -p(-1) = 0$ and $p(-\sqrt{2}) = -p(\sqrt{2}) = 0$ implying 1 and $-\sqrt{2}$ are also roots. Since $p(x)$ is of degree 5 with leading coefficient 1, there is a unique polynomial satisfying the given conditions, namely

    $$p(x) = x(x - 1)(x + 1)(x - \sqrt{2})(x + \sqrt{2}) = x(x^2 - 1)(x^2 - 2).$$

    Thus, $p(2) = 2(2^2 - 1)(2^2 - 2) = 12$. The answer is (C).

11. A set S of five different real numbers is given. When each pair of these numbers are added, you get the ten sums $-4, -3, -2, -1, 0, 1, 4, 5, 6, 8$. The ratio of the largest number in S to the smallest number in S is

(A) $-\frac{14}{3}$ (B) $-\frac{13}{5}$ (C) $-\frac{12}{7}$ (D) $-\frac{5}{7}$ (E) none of these

Solution:
Let $x_1 \le x_2 \le x_3 \le x_4 \le x_5$ be the numbers. Then $x_1 + x_2 = -4, x_1 + x_3 = -3, x_4 + x_5 = 8, x_3 + x_5 = 6$. On the other hand

$$4(x_1 + x_2 + x_3 + x_4 + x_5) = -4 - 3 - 2 - 1 + 0 + 1 + 4 + 5 + 6 + 8 = 14 \iff x_1 + x_2 + x_3 + x_4 + x_5 = \frac{7}{2}$$

Hence $x_3 = -\frac{1}{2}$ and also $x_1 = -\frac{5}{2}, x_2 = -\frac{3}{2}, x_4 = \frac{3}{2}, x_5 = \frac{13}{2}$ and thus $\frac{x_5}{x_1} = -\frac{13}{5}$. The answer is (B).

12. Let $a_n = n^2 + 2n + 50, n = 1, 2, \ldots$. Let d_n be the largest positive integer that is a divisor of both a_n and a_{n+1}. The maximum value of $d_n, n = 1, 2, \ldots$ is

(A) 1   (B) 185   (C) 197   (D) 203   (E) 215

Solution:
d_n must also divide $a_{n+1} - a_n = 2n + 3$, and since $4a_n = 4n^2 + 8n + 200 = (2n + 3)^2 - 2(2n + 3) + 197$, d_n must divide 197. Hence, the largest value of d_n is 197, which is obtained for $n = 97$. The answer is (C)

13. There are seven children belonging to three families. The first family has three children, and the other two families have two children each. In how many ways can these seven children be arranged in a circle so that no two children from the same family are next to each other? Notice that seating arrangements that differ by a rotation are considered the same.

(A) 12   (B) 24   (C) 48   (D) 96   (E) none of these

Solution:

Let A_1, A_2, A_3 be the children of the first family, B_1, B_2 of the second family and C_1, C_2 of the third family. Arrange A_1, A_2, A_3 in a circle with two unoccupied spots between A_1 and A_2, one unoccupied spot between A_2 and A_3 and one unoccupied spot between A_3 and A_1.
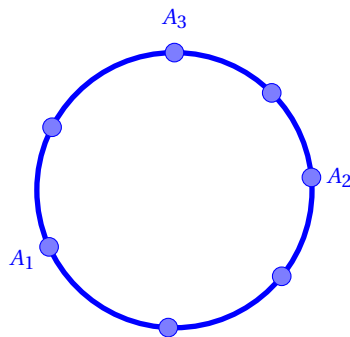
The two free spots between A_1 and A_2, in the direction from A_1 toward A_2, can be occupied by the components of each of the following ordered pairs:

$$(B_1, C_1), (B_1, C_2), (B_2, C_1), (B_2, C_2)(C_1, B_1), (C_1, B_2), (C_2, B_1), (C_2, B_2).$$

For each pair that occupies the first two spots there are two possibilities to fill in the last two unoccupied spots. Therefore there are 16 possibilities to fill in the unoccupied spots. Since there are $3! = 6$ ways to place the three-children family, it comes out that the children can be arranged in $16 \times 6 = 96$ ways. The answer is (D).

14. The positive integers from 1 to 999 are written in a row (with no spaces) in increasing order:

$$12345\cdots 998999.$$

How many times does the sequence of characters "31" occur?

(A) 20   (B) 21   (C) 22   (D) 30   (E) 31

Solution:
For the first 100 positive integers the sequence "31" occurs twice, once from 13 to 14 and again as 31. Now, let x be a three digit number and suppose "31" appears in $x(x+1)$ with the "3" appearing in either the ones, tens or hundreds place of x. If $x = \_\_3$ then $x+1 = \underline{1}\_\_$ which implies $x = \underline{1}\,\underline{a}\,\underline{3}$. Each of $a \in \{0, 1, \ldots, 9\}$ is possible. If $x = \_3\_$ then we must have $x = \underline{a}\,\underline{3}\,\underline{1}$. Each of $a \in \{1, 2, \ldots, 9\}$ is possible. Finally, if $x = 3\_\_$ then we must have $x = \underline{3}\,\underline{1}\,\underline{a}$. Each of $a \in \{0, 1, \ldots, 9\}$ is possible. Thus, the total number of occurrences of "31" is $2 + 10 + 9 + 10 = 31$. The answer is (E).

15. In triangle ABC, D is a point on BC such that $\angle BAD = 30°$ and $\angle DAC = 15°$. If $AB = 3\sqrt{2}$ and $AC = 6$, the length of AD is

(A) $2\sqrt{6}$ (B) $\frac{7\sqrt{2}}{2}$ (C) 5   (D) $3\sqrt{3}$ (E) none of these
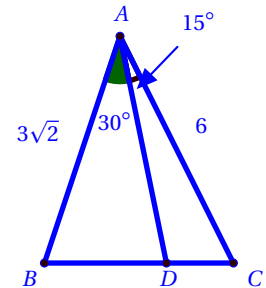
Solution:

Since

$$\text{Area}[ABD] + \text{Area}[ADC] = \text{Area}[ABC]$$

then

$$3\sqrt{2} \cdot AD \cdot \sin 30° + AD \cdot 6 \cdot \sin 15° = 18\sqrt{2}\sin 45°.$$

Notice that

$$\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \sin 30° \cos 45° = \frac{\sqrt{6} - \sqrt{2}}{4}.$$

Hence,

$$AD = \frac{18\sqrt{2} \cdot \frac{\sqrt{2}}{2}}{3\sqrt{2} \cdot \frac{1}{2} + 6 \cdot \frac{\sqrt{6} - \sqrt{2}}{4}} = 2\sqrt{6}.$$

The answer is (A).

16. Let $k \geq 1$ be an integer and $n_k = \lfloor \frac{2^k + 20182019}{2^{k+1}} \rfloor$ where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. The value of the sum $n_1 + n_2 + n_3 + \cdots$ is

(A) 10091008   (B) 10091009   (C) 10091010   (D) 10091011   (E) none of these

Solution:
Notice that for large enough k, $n_k = 0$ and thus only a finite number of terms in the sum $n_1 + n_2 + n_3 + \cdots$ are not equal to 0. It can be also mentioned that $\lfloor x \rfloor$ counts the number of positive integers m such that $m \leq x$. Therefore, the requested sum counts the number of positive integer solutions of

$$n \leq \frac{2^k + 20182019}{2^{k+1}}, \ k \geq 1 \iff 2^k(2n - 1) \leq 20182019, \ k \geq 1.$$

Since every positive even integer can be written uniquely as a product of 2^k, for some integer $k \geq 1$, and an odd positive integer, the sum is equal to the number of positive even integers which are less than or equal to 20182019, which is 10091009. The answer is (B).