The Alberta High School Mathematics Competition

Part II, February 5th , 2020

Problem 1
Let $P(x)$ be a polynomial with integer coefficients. Show that if $P(\frac{1}{3})$ is an even integer then $P(3)$ will also be an even integer.

Problem 2
Find all functions $f(x) = \frac{47}{ax+b}$, where a and b are integers and $a > 0$, so that $f(4)$ and $f(7)$ are both integers.

Problem 3
Consider all the subsets of $\{5, 6, 7, \ldots, 15\}$ having at least two elements. How many of these subsets have the property that the sum of the smallest and the largest element in the subset is 20?

Problem 4
$\triangle ABC$ has right angle at A. Point D lies on AB, between A and B, such that $3\angle ACD = \angle ACB$ and $BC = 2BD$. Find the ratio $\dfrac{DB}{DA}$.

Problem 5
Let $b_0 < c_0$ be real numbers so that the polynomial $f_0(x) = x^2 + b_0 x + c_0$ has two real roots $b_1 < c_1$ (that is, $f_0(b_1) = f_0(c_1) = 0$) and let $f_1(x) = x^2 + b_1 x + c_1$. If $f_1(x)$ has two real roots $b_2 < c_2$, a new quadratic polynomial $f_2(x) = x^2 + b_2 x + c_2$ is constructed. The process is continued until the quadratic polynomial $f_{n-1}(x) = x^2 + b_{n-1} x + c_{n-1}$, $b_{n-1} < c_{n-1}$ has two real roots $b_n < c_n$, but $f_n(x) = x^2 + b_n x + c_n$, $n \geq 1$ has no real roots.

(a) Show that $n \leq 3$.

(b) Show that $n = 3$ is a possible value.